

THE AXIOM OF COHOLOMORPHIC $(2n + 1)$ -SPHERES IN THE ALMOST HERMITIAN GEOMETRY ¹

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In his book on Riemannian geometry [1] E. Cartan proved a characterization of a real-space-form, using the axiom of planes. There are many results in this direction also for a Kaehler manifold. B.-Y. Chen and K. Ogiue [4] have proved that a Kaehler manifold, which satisfies the axiom of coholomorphic 3-spheres is flat. In this paper we prove a generalization of this theorem for an almost Hermitian manifold.

1. Introduction. Let N be an n -dimensional submanifold of an $2m$ -dimensional almost Hermitian manifold M with Riemannian metric g and almost complex structure J . Let $\tilde{\nabla}$ and ∇ be the covariant differentiations on M and N , respectively. It is well known, that the equation $\alpha(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y$, where $X, Y \in \mathfrak{X}N$, defines a normal-bundle-valued symmetric tensor field, called the second fundamental form of the immersion. The submanifold N is said to be totally umbilical, if $\alpha(X, Y) = g(X, Y)H$ for all $X, Y \in \mathfrak{X}N$, where $H = (1/n)\text{trace } \alpha$ is the mean curvature vector of N in M . In particular, if α vanishes identically, N is called a totally geodesic submanifold of M .

For $X \in \mathfrak{X}N$, $\xi \in \mathfrak{X}N^\perp$ we write $\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi$, where $-A_\xi X$ (respectively, $D_X \xi$) denotes the tangential (resp. the normal) component of $\tilde{\nabla}_X \xi$. A normal vector field ξ is said to be parallel, if $D_X \xi = 0$ for each $X \in \mathfrak{X}N$.

By an n -plane we mean an n -dimensional linear subspace of a tangent space. A $2n$ -plane (respectively an n -plane) α where $1 \leq n \leq m$ is said to be holomorphic (respectively, antiholomorphic) if $J\alpha = \alpha$ (respectively $J\alpha \perp \alpha$). A $(2n + 1)$ -plane α is called coholomorphic if it contains a holomorphic $2n$ -plane.

An almost Hermitian manifold M is said to satisfy the axiom of holomorphic $2n$ -planes (respectively $2n$ -spheres) if for each point $p \in M$ and for any $2n$ -dimensional holomorphic plane π in $T_p M$ there exists a totally geodesic submanifold N (respectively a totally umbilical submanifold N with nonzero parallel mean curvature vector) containing p , such that $T_p N = \pi$, where n is a fixed number, $1 \leq n < m$.

An almost Hermitian manifold M is said to satisfy the axiom of antiholomorphic n -planes (respectively n -spheres) if for each point $p \in M$ and for any n -dimensional antiholomorphic plane π in $T_p M$ there exists a totally geodesic submanifold N (respectively a totally umbilical submanifold N with nonzero parallel mean curvature vector) containing p , such that $T_p N = \pi$, where n is a fixed number, $1 < n \leq m$.

An almost Hermitian manifold is called an RK -manifold, if $R(X, Y, Z, U) = R(JX, JY, JZ, JU)$ for all $X, Y, Z, U \in T_p M$, $p \in M$.

We have proved in [5]:

¹SERDICA Bulgaricae mathematicae publicationes. Vol. 8, 1982, p. 391-394

Theorem A. *Let M be a $2m$ -dimensional almost Hermitian manifold, $m \geq 2$. If M satisfies the axiom of holomorphic $2n$ -planes or the axiom of holomorphic $2n$ -spheres for some n , $1 \leq n < m$, then M is an RK-manifold with pointwise constant holomorphic sectional curvature.*

Theorem B. *Let M be a $2m$ -dimensional almost Hermitian manifold, $m \geq 2$. If M satisfies the axiom of antiholomorphic n -planes or the axiom of antiholomorphic n -spheres for some n , $1 < n \leq m$, then M is an RK-manifold with pointwise constant holomorphic sectional curvature and with pointwise constant antiholomorphic sectional curvature. Consequently, if $m \geq 3$, then M is one of the following:*

- 1) a real-space-form, or
- 2) a complex-space-form.

These theorems generalize some results in [3, 6, 9]. It is not difficult to see that if $n > 1$ then the holomorphic analogue of Theorem B holds.

Following B.-Y. Chen and K. Ogiue [4], L. Vanhecke formulates the following axiom of coholomorphic $(2n + 1)$ -spheres [8]:

For each point $p \in M$ and for each coholomorphic $(2n + 1)$ -plane π in $T_p M$, there exists a $(2n + 1)$ -dimensional totally umbilical submanifold N of M containing p , such that $T_p N = \pi$, where n is a fixed integer, $1 \leq n < m$.

We shall prove the following theorem.

Theorem. *Let M be a $2m$ -dimensional almost Hermitian manifold, $m \geq 2$. If M satisfies the axiom of coholomorphic $(2n + 1)$ -spheres for some n , then M is conformal flat.*

Hence, using [7] we have

Corollary 1. *Let M be a $2m$ -dimensional connected Kaehler manifold, $m \geq 2$. If M satisfies the axiom of coholomorphic $(2n + 1)$ -spheres for some n , then either M is flat or M is locally a product of two 2-dimensional Kaehler manifolds with constant curvature K and $-K$, respectively, $K > 0$.*

The case $m \geq 3$ in corollary 1 is treated in [4].

An almost Hermitian manifold M which satisfies $(\tilde{\nabla}_X J)X = 0$ for all $X \in \mathfrak{X}M$ is said to be an NK-manifold. Using the classification in [7] we have also

Corollary 2. *Let M be a $2m$ -dimensional NK-manifold, $m \geq 2$. If M satisfies the axiom of coholomorphic $(2n + 1)$ -spheres for some n , then M is one of the following:*

- 1) a flat Kaehler manifold,
- 2) locally a product $M_1 \times M_2$, where M_1 (respectively M_2) is a 2-dimensional Kaehler manifold with constant curvature K (respectively $-K$),
- 3) a 6-dimensional manifold of constant curvature $K > 0$,
- 4) locally a product $M_3 \times M_2$, where M_3 is a 6-dimensional NK-manifold of constant curvature $K > 0$.

An almost Hermitian manifold M is said to be of pointwise constant type α , provided that for each point $p \in M$ and for each $X \in T_p M$ we have $\alpha(p)g(X, X) = \lambda(X, Y) = \lambda(X, Z)$ with $\lambda(X, Y) = R(X, Y, Y, X) - R(X, Y, JY, JX)$ whenever the planes defined by

X, Y and X, Z are antiholomorphic and $g(Y, Y) = g(Z, Z) = 1$. If for $X, Y \in \mathfrak{X}(M)$ with $g(JX, Y) = g(X, Y) = 0$, $\lambda(X, Y)$ is a constant whenever $g(X, X) = g(Y, Y) = 1$, then M is said to have global constant type.

Corollary 3. *Let M be an almost Hermitian manifold with pointwise constant type α . If M satisfies the axiom of coholomorphic $(2n+1)$ -spheres for some n and if $\dim M \geq 6$, then M is a space of constant curvature α and M has global constant type.*

Corollary 3 is proved in [8] for an RK -manifold.

2. Preliminaries. Let M be an $2m$ -dimensional almost Hermitian manifold with Riemannian metric g , almost complex structure J and covariant differentiation ∇ . The curvature tensor R , associated with ∇ has the following properties:

- 1) $R(X, Y)Z = -R(Y, X)Z$
- 2) $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$
- 3) $R(X, Y, Z, U) = -R(X, Y, U, Z)$

for all $X, Y, Z, U \in T_p M$, $p \in M$, where $R(X, Y, Z, U) = g(R(X, Y)Z, U)$. The Weil conformal curvature tensor C is defined by

$$C(X, Y, Z, U) = R(X, Y, Z, U) - (1/(2m-2))\{g(X, U)S(Y, Z) - g(X, Z)S(Y, U) + g(Y, Z)S(X, U) - g(Y, U)S(X, Z)\} + (S(p)/((2m-1)(2m-2)))\{g(X, U)g(Y, Z) - g(X, Z)g(Y, U)\},$$

where S and $S(p)$ are the Ricci tensor and the scalar curvature of M , respectively.

Now, let N be a submanifold of M , as in section 1. The normal component of $R(X, Y)Z$, where $X, Y, Z \in \mathfrak{X}N$ is given by

$$(2.1) \quad (R(X, Y)Z)^\perp = (\bar{\nabla}_X \alpha)(Y, Z) - (\bar{\nabla}_Y \alpha)(X, Z),$$

where $(\bar{\nabla}_X \alpha)(Y, Z) = D_X \alpha(Y, Z) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z)$ and if N is totally umbilical submanifold of M , (2.1) reduces to

$$(2.2) \quad (R(X, Y)Z)^\perp = g(Y, Z)D_X H - g(X, Z)D_Y H.$$

3. Proof of the Theorem. Let X, Y be arbitrary unit vectors in $T_p M$, $p \in M$, such that X is perpendicular to Y, JY . Applying the axiom of coholomorphic $(2n+1)$ -spheres for a coholomorphic plane, which contains X, JX, JY and is perpendicular to Y and using (2.2) we obtain

$$(3.1) \quad R(X, JX, JY, Y) = 0,$$

$$(3.2) \quad R(JY, JX, X, Y) = 0,$$

$$R(X, JX, JX, Y) = g(D_X H, Y),$$

$$R(X, JY, JY, Y) = g(D_X H, Y).$$

Hence

$$(3.3) \quad R(X, JX, JX, Y) = R(X, JY, JY, Y).$$

From (3.2) we have $R(Y + JY, JX, X, Y - JY) = 0$ and consequently

$$(3.4) \quad R(X, Y, Y, JX) = R(X, JY, JY, JX).$$

If $m > 2$, we take a unit vector Z , perpendicular to X, JX, Y, JY . Using again the axiom of coholomorphic $(2n+1)$ -spheres and (2.2) we find

$$(3.5) \quad R(X, JX, Y, Z) = R(X, Y, JY, Z) = 0 ,$$

$$(3.6) \quad R(X, JX, JX, Z) = R(X, Y, Y, Z) ,$$

$$(3.7) \quad R(X, Y, Y, JX) = R(X, Z, Z, JX) .$$

If $m \geq 4$, let U be a unit vector in $T_p M$, perpendicular to X, JX, Y, JY, Z, JZ . From (3.6) we have $2R(X, JX, JX, U) = R(X, Y+Z, Y+Z, U) = 0$, which gives $R(X, Y, Z, U) = -R(X, Z, Y, U)$.

Hence, by the properties of the curvature tensor R we obtain

$$(3.8) \quad R(X, Y, Z, U) = 0 .$$

Making use of (3.1)-(3.8) it is not difficult to prove that $R(X, Y, Z, U) = 0$ for an arbitrary orthogonal quadruple $X, Y, Z, U \in T_p M$. According to a well known theorem of Schouten [2] the Weil conformal curvature tensor of M vanishes.

Remark. If a Riemannian manifold M of dimension $n > 3$ is conformal flat, then there exists a totally umbilical submanifold N of dimension $n < m$ through every point of M and in every n -dimensional direction of that point (see [2]). Consequently, if M is a conformal flat $2m$ -dimensional almost Hermitian manifold, $m \geq 2$, then M satisfies the axiom of coholomorphic $(2n+1)$ -spheres for every n , $1 \leq n < m$.

R E F E R E N C E S

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